LA-UR -88-358

LA-UR--88-358

DE88 006480

TITLE **FINITE DIMENSIONALITY IN THE COMPLEX GINZBURG-LANDAU EQUATION**

AUTHOR(S) **C. R. Doering, J. D. Gibbon, D. D. Holm, and B. Nicolaenko**

SUBMITTED TO **American Mathematical Society Proceedings**

Los Alamos National Laboratory
Los Alamos, New Mexico 87545

FINITE DIMENSIONALITY IN THE COMPLEX GINZBURG-LANDAU EQUATION

C. R. Doering, J.D. Gibbon, D.D. Holm and B. Nicolaenko

ABSTRACT: Finite dimensionality is shown to exist in the complex Ginzburg-Landau equation

$$A_t = RA + (1+i\nu)A_{xx} - (1+i\mu)A|A|^2$$

periodic on the interval $[0,1]$. A cone condition is derived and explained which gives upper bounds on the number of Fourier modes required to span the universal attractor and hence upper bounds on the attractor dimension itself. In terms of the parameter R these bounds are not large. For instance, when $|\mu| \le \sqrt{3}$, the Fourier spanning dimension is $O(R^{3/2})$. Lower bounds are estimated from the number of unstable side-bands using ideas from work on the Eckhaus instability. Upper bounds on the dimension of the attractor itself are obtained by bounding (or, for $|\mu| \le \sqrt{3}$, computing exactly) the Lyapunov dimension and invoking a recent theorem of Constantin and Foias, which asserts that the Lyapunov dimension, defined by the Kaplan-Yorke formula, is an upper bound on the Hausdorff dimension.

§1 INTRODUCTION. Despite the great advances in understanding the qualitative nature of the various bifurcations which nonlinear systems can undergo, the main results have generally been confined to finite dimensional problems. The infinite dimensional nature of p.d.e's has caused obvious difficulties. The question of whether *ostensibly* infinite dimensional systems can, in fact, behave in a *finite* dimensional way without introducing severe approximations is one which has taxed both applied and pure mathematicians for two decades. It is usual for computational mathematicians to expand a p.d.e in Fourier or other modes and, out of necessity, introduce some form of modal truncation. While they can show success in determining whether a system has apparently stabilised onto a finite number of modes by use of experience and skill, the methods they employ are more on the heuristic level. Until recently, no rigorous proofs have existed which determine whether a system in question is truly finite dimensional. The truncation necessary to integrate a given equation on a machine could easily introduce behaviour which is purely an artefact of that truncation. While a proof of finite dimensionality in the full three dimensional Navier Stokes equations is still beyond us there has been some considerable progress in simpler problems. Indeed, it has been this problem of finite or 'low' dimensionality which has started to engage the minds of analysts in the last two or three years. The two closely interlinked sets of work on the establishment of a *finite* attractor dimension for p.d.e's [1-8] and the work on so-called *inertial manifolds* for the Kuramoto-Sivashinsky (KS) equation [9-11] has stimulated the activity in this area. The KS equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0 \tag{1.1}$$

periodic on $[0\ L]$, has a linear growth rate which is

$$\lambda = k^2 - k^4 \tag{1.2}$$

Discrete modes for which $k^2 > 1$ are stable since $\lambda < 0$. The idea of the so-called 'cone condition' which appears in [9] is simple in physical terms. The system is not in equilibrium since energy is being pumped in through the unstable modes $k^2 < 1$. The stable modes lose energy by diffusion with the energy loss increasing as k increases, as equation (1.2) shows. The nonlinear term pumps energy into the higher modes however, and so the two processes compete. There is however, no *a priori* reason to suppose that a *finite* number of modes can overcome the nonlinear terms in some sense but in fact it turns out that this is so. The aim is to estimate a value of N, derived from a cone condition, which shows when the modes k_n for $n < N$ control the modes for which $n \ge N+1$. This control, which is achieved through a cone condition, slaves the high modes to the low modes by means of a global slaving function Φ. The inertial manifold is essentially the graph of this function Φ: it is a smooth invariant exponentially attracting manifold which contains the universal attractor. On this manifold the p.d.e. is a finite dimensional dynamical system. This idea of slaving an infinite number of degrees of freedom to a finite number of degrees of freedom establishes a direct connection between infinite dimensional problems and the elegant and important work of recent years on finite dimensional dynamical systems.

This paper will concentrate on showing how to construct a cone condition, therefore establishing the existence of an inertial manifold for the complex Ginzburg-Landau (CGL) equation on periodic boundary conditions on [0,1]:

$$A_t = RA + \beta A_{xx} + \gamma A|A|^2 \qquad \gamma,\ \beta \in \mathbb{C} \tag{1.3}$$

This is a simple example of an equation which has a good physical pedigree which can be used as an archetype for understanding how p.d.e's which possess rotating wave solutions behave. It too has a bandwidth of unstable modes since the growth rate of rotating waves solutions is

$$\lambda = R - k^2 \tag{1.4}$$

Hence all modes $k^2 < R$ are unstable and $k^2 > R$ are stable. There is a gap in the spectrum

$$\lambda_n - \lambda_{n+1} = (2n+1)k_1^{\ 2} \tag{1.5}$$

which increases linearly with n. Clearly therefore it is an ideal candidate for an inertial manifold calculation.

The CGL equation occurs as an amplitude equation on long space and time scales in a variety of problems and was first derived by Newell and Whitehead [14] using the finite bandwidth concept. One set of circumstances is in Taylor vortex flow and convection [12,13,15] where the coefficients β and γ are real. For Poiseuille flow [16-18,20] these coefficients are complex. In the latter case $\gamma_r > 0$ and the equation blows up in finite time [17-19]. There are, however, two cases when $\gamma_r < 0$ [20]. The complex Ginzburg-Landau equation also occurs in studies in chemical turbulence [21-23]. The stability of rotating wave solutions to sidebands has been studied widely [24-27] and various papers have investigated numerically the possibility of chaos and pattern formation [28-33]. Since the intention of this paper is to consider various aspects of finite dimensional behaviour, we shall choose $\gamma_r < 0$. Indeed, it will be easiest if we take the equation in the form where blow up does not occur

$$A_t = RA + (1+i\nu)A_{xx} - (1+i\mu)A|A|^2 \tag{1.6}$$

While the circumstances in which the complex Ginzburg-Landau equation arises in the examples above is often highly complicated [15-18], a [illegible] explanation of why it occurs so frequently in the theory of nonlinear waves can be given in a few lines. Let us consider a linear scalar p.d.e.

$$(1.7) \qquad M\left[\frac{\partial}{\partial t};\ \frac{\partial}{\partial x};\ \mu\right]\phi = 0$$

whose complex dispersion relation is $\omega = \omega(k,\mu)$. M is a smooth function of its three independent variables. μ is a parameter such as, for example, the Reynolds or Rayleigh number. The system is unstable when $\omega_r > 0$ so there is a neutral curve in parameter space $\mu = R(k)$ on which $\omega_r = 0$. One can think of linear, oscillating, neutrally stable waves at the most unstable wavenumber k_c being expressed in the form

$$(1.8) \qquad \phi = A\ exp[i(k_c x - w_c t)] + c.c.$$

Now let us consider the nonlinearity of the system in such a way that we can think of the complex dispersion relation becoming dependent upon the amplitude:

$$(1.9) \qquad \omega = \omega(k,\mu,|A|^2)$$

Situations like this are not unknown, for example in optical fibres, where the refractive index of the medium is amplitude dependent. This is equivalent to solving the p.d.e in a modified model form (for example see [14])

$$(1.10) \qquad M\left(\frac{\partial}{\partial t};\ \frac{\partial}{\partial x};\ \mu;\ \phi^2\right)\phi = 0$$

to take account of the nonlinear terms. If we now introduce a small parameter ε defined by $\varepsilon^2 = \mu - \mu_c$ which shows how far we are above the critical point and then define slow space and time scales $X = \varepsilon(x - c_g t)$ and $T = \varepsilon^2 t$, we can then expand M in a fourfold Taylor series and have A dependent on X and T. Using (1.8) and removing secular terms at $O(\varepsilon^3)$ gives equation (1.3). In general, β and γ will be complex. The number R is now a measure of how far the system in question is above criticality on these length and time scales and is therefore used as the main bifurcation parameter in the problem. The correct space and time scales to use in the CGL equation are the slow X and T variables defined above but in future sections we will revert to the usage of small x and t for simplicity.

The main idea of this paper is to show that the complex Ginzburg-Landau equation is 'low dimensional' in the sense it has a finite Fourier spanning dimension (see §2) and a finite attractor dimension (§4). To achieve these results, we will use the idea of inertial manifolds developed in [9] for calculating the Fourier spanning dimension and the methods developed in [35] (see also [1-8])for calculating the attractor dimension. The work discussed in this paper on both the cone condition and attractor dimension has been developed by the authors of this present paper in [36] and on the attractor dimension in [37]. Following on from this, finite dimensionality has been shown to exist a saturable form of the CGL equation which occurs in the ring laser cavity [38]. The ubiquity of the complex Ginzburg-Landau equation makes its behaviour important in the sense that it would be desirable to know how this equation works before trying the harder and more general systems from which it derives in the first place. In §3 we will show how these results fit in with previous work on the sideband instability [24-27].

§2. THE CONE CONDITION FOR THE CGL EQUATION. The idea of a cone condition, introduced in [9], enables us to estimate the number of Fourier modes needed to span the attractor Following [9] we will firstly try to explain the general ideas behind this and then go on to prove that there is indeed an absorbing set in the Hilbert space $\mathcal{H} = L^2[0,1]$. When we have obtained bounds on the necessary norms we will then establish the estimate of the number of Fourier modes needed to span the attractor. These Fourier modes will be eigenfunctions of the Laplacian so let us suppose that we require N of them. We will define P_N to be a projection operator which projects solutions on the universal attractor onto the first N Fourier modes $|k_n| \le k_N$. $Q_N = I - P_N$ is the infinite dimensional projection onto all other modes $|k_n| \ge k_{N+1}$. The idea of a "cone property" is a means of controlling the high modes by the low modes. Figures 1 and 2 give the idea. Let A and A' be two solutions on the universal attractor. Defining $a = A' - A$ with $p = P_N a$ and $q = Q_N a$, we can construct a cone by means of a quality L which is defined by

(2.1) $\qquad L = \|q\|_2^2 - \|p\|_2^2$.

Inside the cone we define $L > 0$, on the cone $L = 0$ and outside $L < 0$. $\|\cdot\|$ denotes the norm in $\mathcal{H} = L^2[0,1]$. We now note that outside the cone, when $\|q\|_2^2 < \|p\|_2^2$, the high modes are controlled by the low modes since low wave number modes can vanish only when high modes vanish. We can now define a function(see [9]):

(2.2) $\qquad \Phi : P_N(attractor) \subset P_N\mathcal{H} \to Q_N\mathcal{H}$

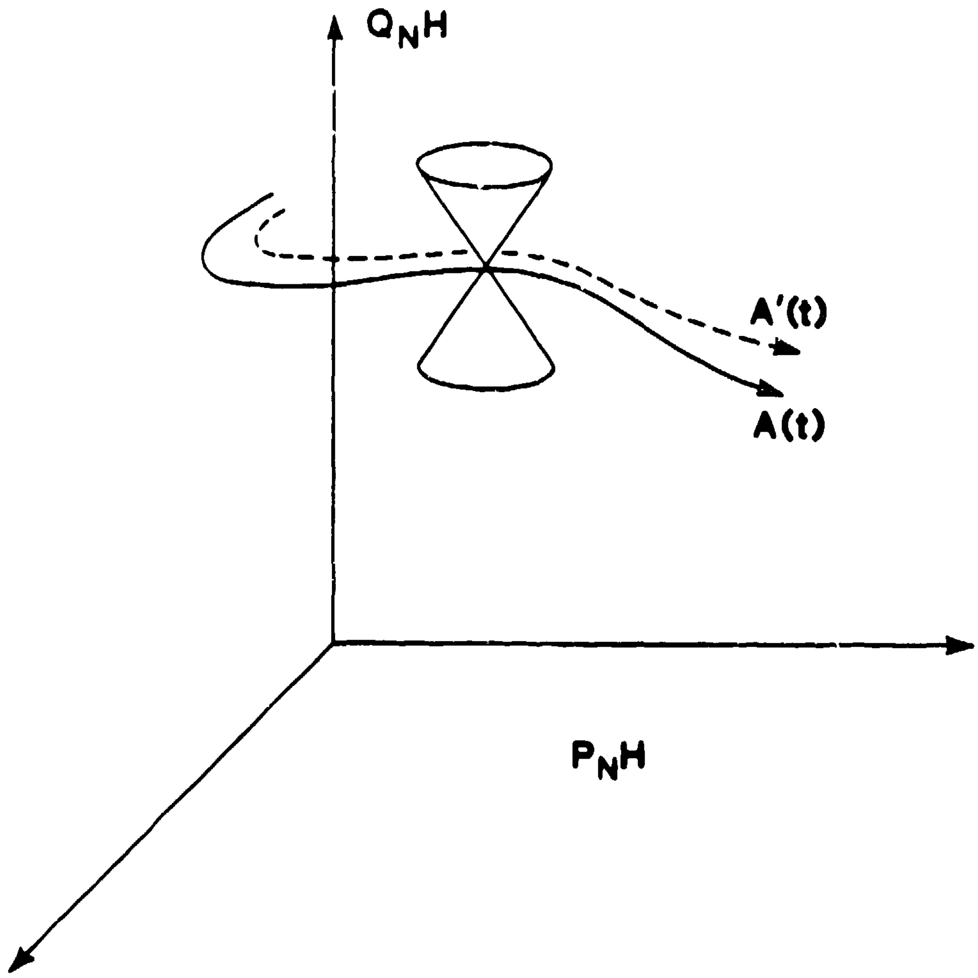

Figure 1: Illustration of the cone condition. At each point along a trajectory A(t) a cone can be drawn and all other trajectories (such as A'(t)) are excluded from the interiors of these cones.

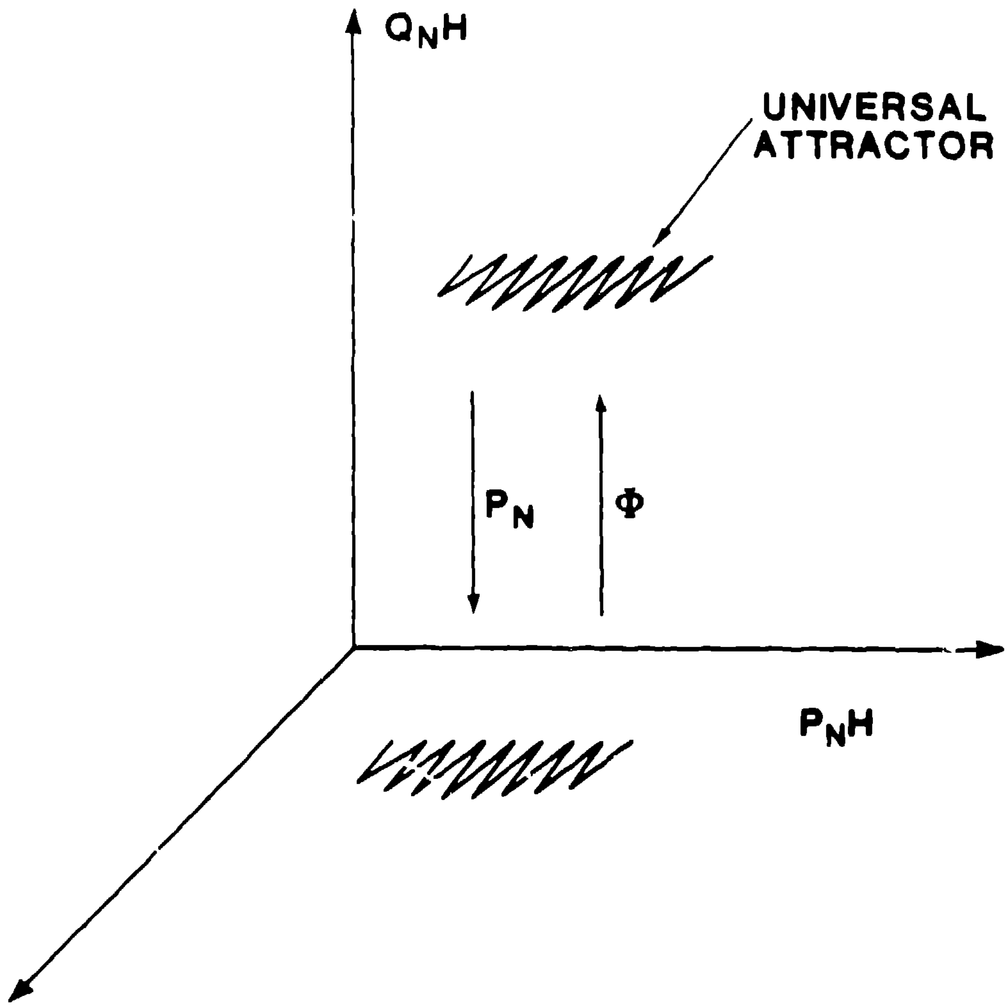

Figure 2: The cone condition on the attractor implies that the projection (P_N) of the attractor onto the finite dimensional space ($P_N H$) is one-to-one onto its image, and hence invertible there ($\Phi = P_N^{-1}$).

which maps a subset of the finite dimensional space $P_N\mathcal{H}$, i.e. $P_N(attractor)$, into the remainder of the configuration space. Hence we can write

$$Q_N A = \Phi(P_N A) \tag{2.3}$$

and the nature of the finite dimensionality can be seen by a slaving of the high modes by the low modes

$$A = \underbrace{P_N A}_{\text{low modes}} + \underbrace{Q_N A}_{\text{high modes}} = P_N + \Phi(P_N A) \tag{2.4}$$

We can at least give a descriptive account of how the cone idea allows a projection which is well defined i.e. (2.2) yields a unique value for Φ. In simplified terms, let us think of two solutions A and A' crossing in configuration space such that they may not touch but there may be at least one point or one which lies vertically above or below a point on the other. At this point $P_N a = 0$ and hence $L > 0$. This can only occur inside the cone by definition and there is no unique projection at these points. However, outside the cone we can project the attractor one to one onto its image. The task is therefore, to find N such that we can make $L < 0$ as $t \to \infty$. If we can achieve this, then asymptotically we can achieve a projection onto N Fourier modes. One can think of the cone construction as a way of 'combing' the attractor as $t \to \infty$ such that a one-to-one projection can be achieved with unprojectable 'entanglements' occurring only inside the cone. An inertial manifold is a smooth (Lipschitz) exponentially attracting invariant manifold which contains the universal (global) attractor. Restricted to this manifold the CGL equation is equivalent to a finite dimensional dynamical system. This projection now allows the universal attractor to be spanned by $2N+1$ Fourier modes. This provides an upper bound on the attractor dimension although this estimate is not necessarily good. With a cone condition, it is possible to use standard invariant manifold techniques, such as centre manifold methods [34] to perform the required extension to an inertial manifold. This suggests the identification of an inertial manifold, which is the graph of the extension over the new domain, as a global centre manifold. The geometric idea of an inertial manifold is given in Figure 3 and can be pursued in references [1-9]

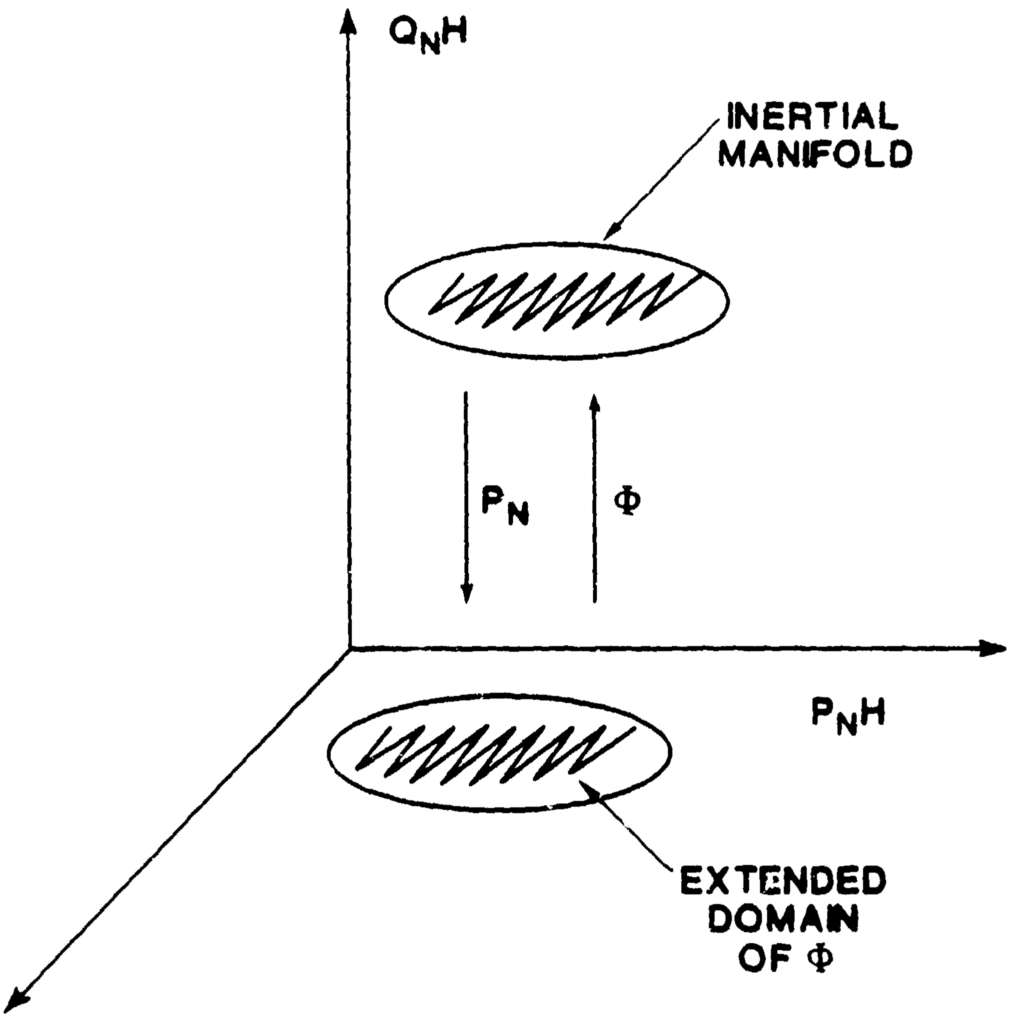

Figure 3: An inertial manifold is the extension of the domain of definition of Φ off of the projection of the attractor, preserving the reduced dynamics. The graph of Φ is an invariant, exponentially attracting manifold in the configuration space that contains the universal attractor.

The idea of a cone condition, explained above, enables us to estimate the number of Fourier modes needed to span the universal attractor but before we explain how to do this, it is necessary to prove that there exists an 'absorbing set'. Let us define a mass $M(t)$ and an energy $E(t)$ by

$$(2.5) \qquad M = \int_0^1 |A|^2 dx \qquad\qquad E = \int_0^1 |A_x|^2 dx .$$

Differentiation of M w.r.t. t and substitution of A_t and A_t^* from the CGL equation gives

$$(2.6) \qquad ½M_t = RM - E - \int_0^1 |A|^4 dx$$

whence the use of the Cauchy-Schwarz inequality gives

$$(2.7) \qquad ½M_t \le RM - E - M^2 \le RM - M^2$$

so

$$(2.8) \qquad M(t) \le R[1-\exp(-2Rt)]^{-1}$$

Hence in our Hilbert space $K = L^2[0,1]$, the flow contracts into a ball of radius $R(\sigma) = [(1+\sigma)R]^{½}$ in a finite time $t(\sigma) = -(\ln[\sigma/1+\sigma])/2R$. In other words, $M \le R$ as $t \to \infty$, for all ν and μ. When we turn to E, a calculation whose details we will not give, we have to make a distinction between $|\mu| \gtrless \sqrt{3}$. We find that

$$(2.9a) \qquad \lim_{t\to\infty} E \le (1+\sigma)^2 R^2 \qquad |\mu| \le \sqrt{3}$$

$$(2.9b) \qquad \lim_{t\to\infty} E \le \delta^2 R^2 (1+[1+\delta^{-2}(1+\delta)/R]^{½})^2 \qquad \mu > \sqrt{3}$$

where

$$(2.9c) \qquad \delta = \max(0, -2+|1+i\mu|)$$

It is now necessary to find a uniform L^∞ norm for $A(x,t)$. These turn out to be, for ν arbitrary (see [36])

$$\lim_{t\to\infty} \|A\|_\infty^2 = \lim_{t\to\infty} \sup_x |A(x,t)|^2$$

$$\le \begin{cases} R + 2R^{3/2} & |\mu| \le \sqrt{3} \qquad (2.10a) \\ R + 2\delta R^2(1+[1+\delta^{-2}(1+\delta)/R]^{½}) & |\mu| > \sqrt{3} \qquad (2.10b) \end{cases}$$

It is also possible to find bounds when $\nu = 0$ and μ arbitrary [36]. Let us now return again to the cone condition. In terms of $a = A' - A$, the difference between two solutions, we have

$$(2.11) \qquad a_t = Ra + (1+i\nu)a_{xx} - (1+i\mu)(A'|A'|^2 - A|A|^2)$$

Operating on this equation with P_N and Q_N to form p and q, we have

$$(2.12a) \quad p_t = Rp + (1+i\nu)p_{xx} - (1+i\mu)P_N[A'|A'|^2 - A|A|^2]$$

$$(2.12b) \quad q_t = Rq + (1+i\nu)q_{xx} - (1+i\mu)Q_N[A'|A'|^2 - A|A|^2]$$

Multiplying (2.12a) by p^*, integrating over $[0,1]$ and taking the real part and repeating the process for (2.12b), we obtain

$$(2.13a) \quad \tfrac{1}{2}\frac{\partial}{\partial t}\|p\|_2^2 = R\|p\|_2^2 + Re\{(1+i\nu)\int_0^1 p^* p_{xx}dx\} - Re\{(1+i\mu)\int_0^1 p^* P_N[A'|A'|^2 - A|A|^2]dx\}$$

$$(2.13b) \quad \tfrac{1}{2}\frac{\partial}{\partial t}\|q\|_2^2 = R\|q\|_2^2 + Re\{(1+i\nu)\int_0^1 q^* q_{xx}dx\} - Re\{(1+i\mu)\int_0^1 q^* Q_N[A'|A'|^2 - A|A|^2]dx\}$$

In order to construct a differential inequality for $L = \|q\|_2^2 - \|p\|_2^2$, we need to estimate not only the nonlinear terms but the terms:

$$(2.14) \quad Re\{(1+i\nu)\int_0^1 [q^* q_{xx} - p^* p_{xx}]dx\} = \int_0^1 (|p_x| - |q_x|^2)dx \quad .$$

However, we know that on periodic boundary conditions

$$(2.15) \quad \int_0^1 |p_x|^2 dx \le k_N^2 \|p\|_2^2; \qquad \int_0^1 |q_x|^2 dx \ge k_{N+1}^2 \|q\|_2^2$$

The right hand side of (2.14) now obeys the inequality

$$(2.16) \quad \int_0^1 (|p_x|^2 - |q_x|^2)dx \le k_N^2 \|p\|_2^2 - k_{N+1}^2\|q\|_2^2$$

Subtracting (2.13a) from (2.13b) and using (2.16) we have

$$(2.17) \quad \tfrac{1}{2}\frac{\partial}{\partial t}(\|q\|_2^2 - \|p\|_2^2) \le (R-k_N^2)(\|q\|_2^2 - \|p\|_2^2) - (k_{N+1}^2 - k_N^2)\|q\|_2^2 + \textit{nonlinear term}$$

Now we can observe how the gap competes with the nonlinear terms

$$(2.18) \quad \tfrac{1}{2}L_t \le (R-k_N^2)L - (2N+1)k_1^2\|q\|_2^2 \quad + \textit{nonlinear term}$$

There are two points here. Firstly, as we explained in §1, *stable* modes occur when $k_N^2 > R$. Hence the sign on the nonlinear term L is negative. If we can bound the nonlinear terms in terms of $\|q\|_2^2$ then we can overcome these terms by choosing N large enough. Bounds on the nonlinear terms can be found in the following way:

$$(2.19) \quad \textit{Nonlinear term} = -Re\{(1+i\mu)\int_0^1 [q^* Q_N - p^* P_N][A'|A'|^2 - A|A|^2]dx\}$$

Using the fact that the projections are self-adjoint, we find

$$(2.20)\qquad \textit{Nonlinear term} = Re\left((1+i\mu)\int_0^1 (q^* - p^*)(A'|A'|^2 - A|A|^2)dx\right)$$

Part of the integrand of (2.20) can be written as

$$(2.21)\qquad A'|A|^2 - A|A|^2 = a(|A'|^2 + |A|^2) + a^* AA' = (p+q)(|A'|^2 + |A|^2) + a^* AA'$$

and we find that

$$(2.22)\qquad \textit{Nonlinear term} = -Re\left\{(1+i\mu)\int_0^1 [-|q|^2 + |p|^2 - 2i\,Im(q^* p)]\right.$$

$$\left.\times (|A'|^2 + |A|^2) + (1+i\mu)\int_0^1 (p^{*2} - q^{*2})AA'\,dx\right\}$$

$$\le 2\|A\|_\infty^2 \|p\|_2^2 + 4\mu\|A\|_\infty^2 \int |q||p|dx$$

$$+ |1+i\mu|(\|p\|_2^2 + \|q\|_2^2)\|A\|_\infty^2$$

$$\le \|A\|_\infty^2 \left[2\|p\|_2^2 + 4\mu\|q\|_2\|p\|_2 + \sqrt{1+\mu^2}\,(\|p\|_2^2 + \|q\|_2^2)\right]$$

Consequently we have

$$(2.23)\qquad \tfrac{1}{2}L_t \le (R - k_N^2)L - (2N+1)k_1^2\|q\|_2^2$$

$$+ \|A\|_\infty^2 \left(2\|p\|_2^2 + 4\mu\|p\|_2\|q\|_2 + \sqrt{1+\mu^2}\,(\|p\|_2^2 + \|q\|_2^2)\right)$$

Inside the cone, when $\|q\|_2^2 > \|p\|_2^2$ $(L > 0)$ we can choose N large enough such that

$$(2.24)\qquad \tfrac{1}{2}L_t < (R - k_N^2)L$$

and so L is decreasing as $t \to \infty$ for those modes (stable modes) for which $k_N^2 > R$. When L reaches zero on the cone, we find that $L_t < 0$ and to L decreases further to become negative. Once negative, it must remain negative. In summary therefore, if we choose

$$(2.25)\qquad 2\pi N_{max} > max\left(R^{\frac{1}{2}},\ \|A\|_\infty^2 \left[1 + \sqrt{1+\mu^2} + 2|\mu|\right]/k_1 - \tfrac{1}{2}k_1\right)$$

where k_1 is the fundamental wave number, then trajectories inside the absorbing set which are inside the cone are expelled from the cone in an exponential way as $t \to \infty$ and once outside remain outside. We note that we have the estimates for $\|A\|_\infty^2$ in equation (2.10). This estimate is an *upper bound* on the attractor dimension uniform in ν. Equation (2.10) shows that this upper bound goes like $R^{1/2}$ for $|\mu| \le \sqrt{3}$ and like R^2 for

$|\mu| > \sqrt{3}$. In fact in [36], we find that N_{max} goes like R when $\nu = 0$ for all μ. These bounds are obviously dependent upon the estimates for $\|A\|_\infty^2$

§3. THE MODULATIONAL INSTABILITY: LOWER BOUNDS ON THE FOURIER SPANNING AND ATTRACTOR DIMENSIONS.

The CGL equation is well known to possess so-called rotating wave solutions. These must be contained in the universal attractor - indeed they form a lower bound on both the Fourier spanning dimension (D) and the universal attractor dimension (d). These solutions take the form

(3.1a) $\quad A_n = a_n \exp[i(k_n x - \omega_n t)] \qquad n = 0, 1, 2, \ldots$

(3.1b) $\quad \omega_n = R\nu + (\mu+\nu)k_n^2$

(3.1c) $\quad |a_n|^2 = R^{½} = k_n^2$

The first set of bifurcation points are $R^{½} = k_n^2$ $(k_n = \pm 2\pi n)$ for each n, including $n = 0$, which grow off the zero solution. Although these solutions appear trivial, they are expressed in terms of Fourier modes and so they give a lower bound on the Fourier spanning dimension

(3.2) $\quad D \geq D_{rot} = 1 + 2[R^{½}/2\pi]$

Each Fourier mode k_n may be stable or unstable to a neighbour k_m. To test for this we use the method described in Stuart & DiPrima [26] for discussing the Eckhaus instabilty [24-25]. This requires us to look at the stability of A_n in the following way. We write

(3.3) $\quad A = A_n(x,t)[1 + B(x,t)]$

We now substitute this into the CGL equation

(3.4) $\quad A_t = RA + (1+i\nu)A_{xx} - (1+i\mu)A|A|^2$

and linearise in B. This gives

(3.5) $\quad B_t = (1+i\nu)B_{xx} + 2i(1+i\mu)k_n B_x - (1+i\mu)|a_n|^2(B+B^*)$

To test for the stability of a wavenumber k_n against another k_m, we write B as

(3.6) $\quad B = p^+(t)\exp(ik_m x) + p^-(t)\exp(-ik_m x)$

and obtain for p^+, p^-

(3.7a) $\quad \dfrac{d}{dt}\begin{pmatrix} p^{+*} \\ p^- \end{pmatrix} = \begin{pmatrix} -c^{+*} & -(1-i\mu)|a_n|^2 \\ -(1+i\mu)|a_n|^2 & -c^- \end{pmatrix}\begin{pmatrix} p^{+*} \\ p^- \end{pmatrix}$

(3.7b) $\quad$ where $c^{\pm} = (1+i\nu)k_m^2 \pm 2(1+i\nu)k_n k_m + (1+i\mu)|a_n|^2$

Neutral curves can be found for the Hopf bifurcation in (3.7) which occur when

$$(3.8a)\qquad 4k_n^2 = \frac{(|a_n|^2+k_m^2)^2[(1+\nu^2)k_m^2+2\epsilon|a_n|^2]}{(|a_n|^2+k_m^2)^2 + (\mu|a_n|^2 + \nu k_m^2)^2}$$

$$(3.8b)\qquad \epsilon = 1+\mu\nu$$

$$(3.8c)\qquad |a_n|^2 = R-k_n^2$$

The sign of ϵ is important since it determines the asymptotic direction of the neutral curves in the k_n^2 versus R plane. In the following two diagrams, if a point in the $(R^{1/2}, |k_n|)$ plane lies *above* the mth curve then the nth rotating wave is *unstable* to the mth sideband and if the point lies *below*, it is *stable*. Figure 4 is plotted for values $\mu = -\sqrt{3}$, $\nu = -30\sqrt{3}$ so that $\epsilon = 91$ and Figure 5 has values $\mu = -\sqrt{3}$, $\nu = +30\sqrt{3}$ so $\epsilon = -89$.

The general features are as follows. When ν is large and negative the curves are bunched up along the diagonal and each rotating wave becomes linearly stable as R is increased. This region ($\epsilon > 0$) is known as the modulationally stable regime [25] since the $k = 0$ solution is always stable. Our calculations here generalise the results in [25] which was based solely on stability of the $k = 0$ mode (spatially homogeneous states) - our results here are valid for any k_n. In the limit of $\nu \to -\infty$ the relation between k_n^2 and R becomes linear

$$(3.9)\qquad k_n^2 = R + 3k_m^2-2k_m(R+4k_m^2)^{1/2}$$

independently of μ. As ν is increased, the curves unfold. When $\mu = \nu$

$$(3.10)\qquad k_n^2 = \frac{1}{3} R + \frac{1}{6} k_m^2$$

again independently of μ.

As ν increases such that ϵ becomes negative, the curves turn over and cut the R-axis. This is now the modulationally unstable regime as each rotating wave will eventually become unstable to any k_m as R is increased. As ν is increased the intersection points through the R-axis get closer to the origin. Their point of closest approach occurs at

$$(3.11)\qquad \nu = -[1+\sqrt{(1+\mu^2)}]/\mu$$

As ν is increased past this value the intersection points start to move away from the origin. It is at the value of ν given in (3.11) that the spatially homogeneous rotating wave solution ($k = 0$) has its maximum number of unstable directions. Other remarks about the rotating waves

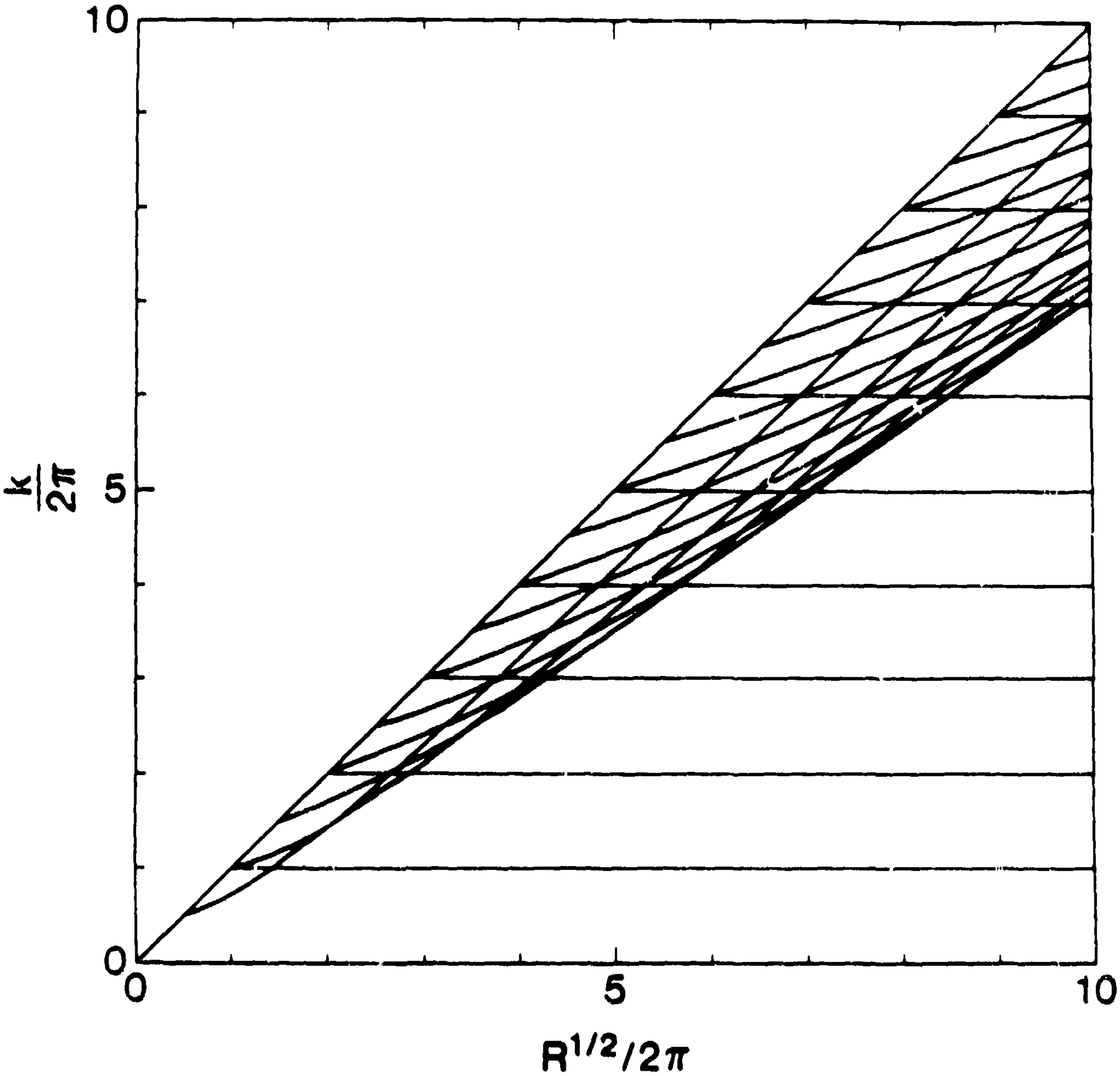

Figure 4: Neutral stability curve in the $(k, \sqrt{R})$ plane for parameter values $\mu = -\sqrt{3}$, $\nu = -30\sqrt{3}$.

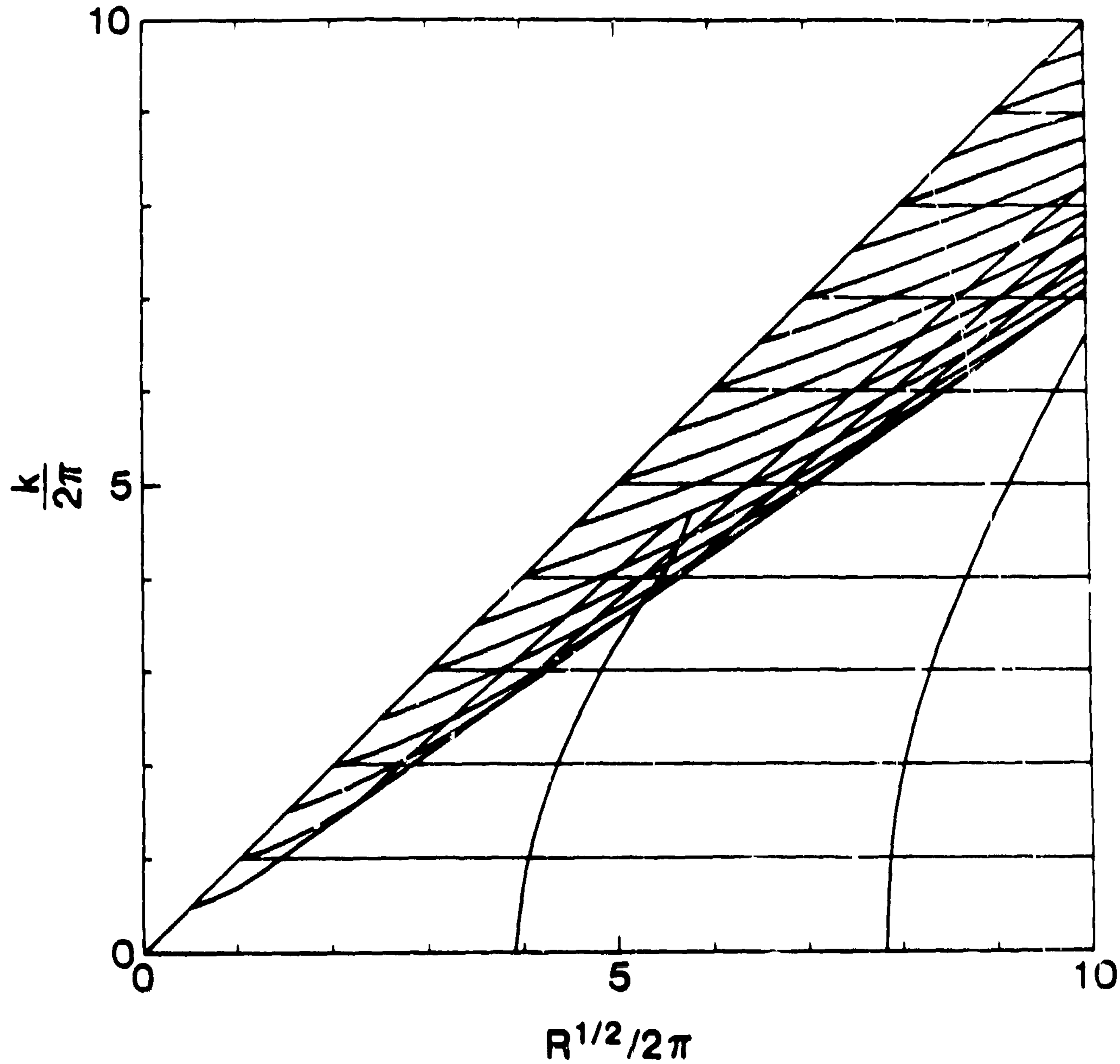

Figure 5: Neutral stability curve in the $(k, \sqrt{R})$ plane for $\mu = -\sqrt{3}$, $\nu = 30\sqrt{3}$.

can be found in [36].

The value of R at which the spatially homogeneous rotating wave becomes unstable to side-band perturbations of wavenumber k_m is

$$(3.12) \qquad R = -k_m^2(1+\nu^2)/2\epsilon$$

We now proceed with the task of this section, to develop lower bounds on the attractor and Fourier spanning dimensions. First, for the attractor dimension, we note that the linearized stability analysis (including neutral modes) gives the dimension of the unstable manifold of each of the rotating wave solutions. For each specific set of the parameters μ and ν, this set of dimensions for the rotating wave solutions can be determined as a function of R simply by counting, using the plot of neutral stability curves. For this counting it must be noted that anywhere in the physically relevant region in the (R,k^2) plane, i.e. $R \geq k^2$, the sum of the real parts of the two eigenvalues of the matrix in (2.6) is negative:

$$(3.13) \qquad Re(\lambda_1) + Re(\lambda_2) = -2(|a_n|^2+k_m^2) < 0 .$$

Thus, at least one of the eigenvalues has a negative real part. This means that when a rotating wave lies in the unstable region of a particular side-band it has only a *one* dimensional unstable direction associated with that side-band. Hence, in the counting procedure we add *one* to the unstable manifold dimension for each unstable side-band.

The dimension $D_{rot}(R)$ of the unstable manifold for the trivial solution $A = 0$ is given in equation (3.2). Although in any particular situation the counting procedure described above can be carried out for any specific rotating wave and set of parameters, there is no convenient explicit expression for the dimension of the unstable manifold of an arbitrary rotating wave solution. However, the spatially homogeneous rotating wave is amenable to computation, and the maximum of its and the trivial solution's unstable manifold dimensions provides a lower bound on the universal attractor dimension. We will use this lower bound in what follows.

In the modulationally unstable case ($\epsilon < 0$) the number of unstable directions for the spatially homogeneous rotating wave is computed from (3.12)

$$(3.14) \qquad m = [(-2\epsilon/(1+\nu^2))^{1/2}(R^{1/2}/2\pi)]$$

Since the rotating wave itself is one-dimensional, a lower bound on the universal attractor dimension is

$$(3.15)\qquad d \ge max(1+[(-2\epsilon/(1+\nu^2))^{1/2} R^{1/2}/2\pi)],\ D_{rot}(R)) .$$

We may bound this lower bound from above, uniformly in ν, by recalling that the closest point of approach of the intersections of the neutral curves and the R axis to the origin occurs for $\nu = -(1+(1+\mu^2)^{1/2})/\mu$. Hence, a lower bound on the universal attractor dimension is given by

$$(3.16)\qquad \sup_{\nu} d \ge max(1+[(\mu^2/(1+(1+\mu^2)^{1/2}))^{1/2}(R^{1/2}/2\pi)],\ D_{rot}(R)) .$$

For $|\mu| \le \sqrt{3}$, the second term in (3.16) is always greater than or equal to the first. When $|\mu| > \sqrt{3}$, the first term can dominate the second for some values of R, since $\sqrt{3}$ is the critical value of μ for which there exists a ν such that the homogeneous rotating wave is unstable to a side-band *before* the trivial solution goes unstable to the second ($k = 2\pi$) rotating wave as R is increased from 0. When $|\mu|$ is larger than $\sqrt{24}$ the first term in (3.16) is always larger than the second. We summarise these lower bounds on the universal attractor dimension, uniform in ν, as follows:

$$(3.17a)\qquad \sup_{\nu} d \ge 1+2[R^{1/2}/2\pi] ,\qquad |\mu| \le \sqrt{3} ,$$

$$(3.17b)\qquad \sup_{\nu} d \ge max(1+[(\mu^2/(1+(1+\mu^2)^{1/2}))(R^{1/2}/2\pi)],\ D_{rot}(R)) ,\qquad \sqrt{3} < |\mu| \le \sqrt{24} ,$$

$$(3.17c)\qquad \sup_{\nu} d \ge 1+[(\mu^2/(1+(1+\mu^2)^{1/2}))^{1/2}(R^{1/2}/2\pi)],\qquad \sqrt{24} < |\mu| .$$

We remark that the border-line situation $|\mu| = \sqrt{3}$ has appeared in the previous section in the nonlinear analysis.

Next, to obtain lower bounds on the Fourier spanning dimension D we utilize some different features of the neutral stability curves obtained from the side-band stability analysis. As before, we could take the pedestrian route for each value of the parameters and simply count, but we seek analytic expressions - and especially some evidence of uniformity in the parameters. Along the diagonals in Figures 3 and 4, where the rotating waves come into existence, the intersections of the neutral curves occur in a regular way: setting $|a_n|^2 = 0$ in (2.8) we have $k_n^2 = k_m^2/4$. Thus, when the nth rotating wave comes into existence, it is unstable (or neutral) to the first $2n$ side-bands. It immediately, stabilizes to the 2nth side-band, although it can go unstable to that side-band again as R is increased. Away from the diagonals in Fig. where $|a_n|^2 > 0$ the matrix in (3.7) is not diagonal, so it takes both Fourier modes ($\pm k_m$) to span the relevant unstable direction. An estimate on a lower bound for the Fourier spanning dimension - really an upper envelope on any lower bound obtained this way - is then given by

$$(3.18) \quad \sup_{\mu,\nu} D \approx > 1+3[R^{1/2}/2\pi] \ ,$$

where the 3 comes from the fact that the highest wavenumber that supports an unstable direction of the nth rotating wave at $R = k_n{}^2$ is $2\pi(n+2n)$: this count includes both the number of Fourier modes required to span all the rotating waves, as well as those required to span their unstable directions. This count is a strict lower bound for each μ and ν at the points where $R^{1/2}/2\pi$ is an integer and it is convenient for its uniformity in the parameters.

The expression (3.18) is not an accurate estimate deep in the modulationally unstable regime where a rotating wave can be unstable to many side-bands. An improved estimate is obtained by taking over the the results above for the dimension of the unstable manifold of the $k = 0$ rotating wave. Since there are in general two Fourier modes required to span each unstable dimension, one may also invoke the rigorous lower bounds

$$(3.19a) \quad \sup_{\nu} D \geq 1+2[R^{1/2}/2\pi] \ , \qquad |\mu| \leq \sqrt{3} \ ,$$

$$(3.19b) \quad \sup_{\nu} D \geq 1+2[((\mu^2/(1+\mu^2)^{1/2}))^{1/2}(R^{1/2}/2\pi)] \ , \qquad |\mu| > \sqrt{3} \ .$$

§4. THE KAPLAN-YORKE FORMULA: UPPER BOUNDS ON THE ATTRACTOR DIMENSION.

In §2 we used the cone condition to find an upper bound on the Fourier spanning dimension. This, of course, provides an upper bound on the attractor dimension although it is not necessarily a good upper bound. We would expect that direct computation of the attractor dimension through the Kaplan-Yorke formula [35] should provide better estimates. The central result we will use is the theorem of Constantin & Foias [1] which asserts that the Lyapunov dimension (d_L) defined in [35] with the global Lyapunov exponents, is an upper bound on the Hausdorff dimension (d_H) of the attractor. We will briefly review the concepts of fractal and Lyapunov exponents, the Kaplan-Yorke formula and provide a heuristic justification of the theorem of Constantin & Foias in terms of the fractal dimension d_F. It will turn out that when we compute d_L exactly in the case $|\mu| \le \sqrt{3}$, the lower and upper bounds coincide, therefore giving the attractor dimension exactly. We will also provide rigorous upper bounds on d_L for $|\mu| > \sqrt{3}$.

The fractal dimension of a bounded set in a metric space extends the usual notion of dimension in euclidean space. Let $N(\varepsilon)$ denote the minimum number of balls of radius $\varepsilon > 0$ required to cover a compact set. For a set of an integer dimension d, the number $N(\varepsilon)$ is proportion to ε^{-d} for small ε. The fractal dimension (which need not be an integer) is defined by

$$(4.1) \qquad d_F := \lim_{\varepsilon \downarrow 0} \log(N(\varepsilon))/\log(1/\varepsilon) \quad .$$

Hence d_F is the scaling exponent for the "volume" displaced by the set on a scale ε.

The Lyapunov exponents on the attractor of a dynamical system describe the exponential rate of divergence of trajectories that start at nearby points. A positive Lyapunov exponent indicates sensitive dependence on initial conditions and hence chaos, for in this case the prediction of a trajectory after a long time requires an exponentially accurate knowledge of its initial conditions. To compute the global Lyapunov exponents, one considers the linearized flow along a trajectory on the universal attractor. For a solution $A(t) \in X$ of the CGLE

$$(4.2) \qquad \partial_t A = RA + (1+i\nu)A_{xx} - (1+i\mu)|A|^2A \ ,$$

the linearized flow of the vector $\xi(t) \in X$, along $A(t)$, is defined by

$$(4.3) \qquad \partial_t \xi = R\xi + (1+i\nu)\xi_{xx} - 2(1+i\mu)|A|^2\xi - (1+i\mu)A^2\xi^* \ .$$

We denote the solution $\xi(t)$ of (4.3) as $L(t,A_0)\xi$, where ξ is the initial condition and A_0 is the initial condition for the solution of the nonlinear flow (4.2). The first global Lyapunov exponent is the rate of asymptotic exponential growth of the length of a vector developing according to (4.3), maximized over all solutions $A(t)$ on the universal attractor and all possible directions of ξ:

$$(4.4) \qquad \lambda_1 := \limsup_{t \to \infty} t^{-1} \log(\sup_{A_0} \sup_{\|\xi\| \le 1} \|L(t,A_0)\xi\|)$$

(the supremum over A_0 is restricted to initial conditions on the univesal attractor). The second global Lyapunov exponent is determined by the largest asymptotic growth rate of areas evolving according to the linearized flow:

$$(4.5) \qquad \lambda_1+\lambda_2 := \limsup_{t\to\infty} t^{-1} \log\Big(\sup_{A_0} \sup_{\|\xi_i\|\le 1} \|L(t,A_0)\xi_1 \wedge L(t,A_0)\xi_2\|\Big) \ .$$

Similarly, the sum of the first n global Lyapunov exponents governs the largest exponential growth rates of n-volumes, according to

$$(4.6) \qquad \lambda_1+\ldots+\lambda_n := \limsup_{t\to\infty} t^{-1}\log\Big(\sup_{A_0} \sup_{\|\xi_i\|\le 1} \|L(t,A_0)\xi_1\wedge\ldots\wedge L(t,A_0)\xi_n\|\Big)$$

In equations (4.5-6) above the magnitudes of the volume elements are given by the norms on the corresponding spaces of n-forms on H. That is, for vectors $\xi_i, \xi_j \in H$,

$$(4.7) \qquad \|\xi_1\wedge\ldots\wedge\xi_n\|^2 = \langle\xi_1\wedge\ldots\wedge\xi_n,\ \xi_1\wedge\ldots\wedge\xi_n\rangle \ ,$$

$$\langle\xi_1\wedge\ldots\wedge\xi_n, \zeta_1\wedge\ldots\wedge\zeta_n\rangle = \det M \quad \text{with} \quad M_{ij} = \langle\xi_i,\xi_j\rangle \ ,$$

where $\langle\cdot,\cdot\rangle$ is the inner product of H (in our situation, $\langle\xi,\zeta\rangle = \int_0^1 dx\,\xi^*(x)\zeta(x))$. Loosely speaking an n-volume V_n around the attractor develops according to $V_n(t) = V_n(0)\exp[(\lambda_1+\ldots+\lambda_n)t]$. The Lyapunov exponents obey the ordering $\lambda_1 \ge \lambda_2 \ge \ldots \to \infty$.

The Lyapunov dimension of the attractor, d_L, is defined by the following procedure. Consider the integer m such that

$$(4.8) \qquad \lambda_1 + \ldots + \lambda_m \ge 0, \quad \text{but} \quad \lambda_1 + \ldots + \lambda_{m+1} < 0 \ .$$

Then d_L is defined by the Kaplan-Yorke formula

$$(4.9) \qquad d_L := m + (\lambda_1 + \ldots + \lambda_m)/|\lambda_{m+1}| \ .$$

Note that $\lambda_{m+1} < 0$ and $-\lambda_{m+1} > \lambda_1 + \ldots + \lambda_m$ so that $m \le d_L < m+1$, and d_L is not necessarily an integer. The Lyapunov dimension would seem to give an upper bound on the fractal dimension, in view of the following argument [34,36]. With m chosen according to (4.8), consider a covering of the attractor by $m+1$ dimensional balls of radius $\varepsilon > 0$ (assuming thta this can be done), and let $N'(\varepsilon)$ denote the number of such balls required. Then after a time t, the images of these balls under the nonlinear flow (4.2) will still cover the attractor, but they will be transformed into $m+1$

dimensional ellipsoids of principal axes $\epsilon exp(\lambda_1 t),\ldots,\epsilon exp(\lambda_{m+1} t)$. Ignoring geometrical factors, it then takes on the order of $N'(\epsilon)exp(\lambda+\ldots+\lambda_m-m\lambda_{m+1})t$ smaller balls of radius $\epsilon e\lambda_{m+1}t$ to cover the attractor. Since this is by no means the fewest number of such balls (denoted $N(\epsilon e\lambda_{m+1}t)$) required to cover the attractor, we have

$$(4.10)\qquad N(\epsilon e\lambda_{m+1}t) \le N'(\epsilon)\exp(\lambda_1+\ \ldots\ +\lambda_m-m\lambda_{m+1})t\ .$$

Hence, according to (4.1),

$$\begin{aligned}(4.11)\qquad d_F &\le \lim_{t\to\infty} log(N\epsilon exp(\lambda_{m+1}t)))/log(\epsilon^{-1}exp(-\lambda_{m+1}t))\\ &= \lim_{t\to\infty}[\lambda_1+\ldots+\lambda_m-m\lambda_{m+1}+t^{-1}logN'(\epsilon)]/(-\lambda_{m+1}-t^{-1}log\epsilon]\\ &= d_L\ .\end{aligned}$$

We stress that this argument is merely heuristic and this result has not been proven rigorously. However, Constantin and Foias [1] have rigorously established a related result, i.e. $d_H \le d_L$, even in certain infinite dimensional cases. In general $d_F \ge d_H$, so the reasoning above is consistent with the rigorous result.

We now proceed to the computation (or estimation) of the Lyapunov exponents for CGLE. The linearized time development of an n-form $\xi_1(t)\wedge\ \ldots\ \wedge\xi(t)$ is given by

$$\begin{aligned}(4.12)\qquad (d/dt)\xi_1(t)\wedge\ldots\wedge\xi_n(t) &= (d\xi_1/dt)\wedge\ldots\wedge\xi_n+\ \ldots\ +\xi_1\wedge\ldots\wedge(d\xi_n/dt)\\ &= F(t,A_0)\xi_1(t)\wedge\ldots\wedge\xi_n(t)+\ \ldots\ +\xi_1(t)\wedge\ldots\wedge F(t,A_0)\xi_n(t)\end{aligned}$$

where $F(t,A_0)$ is the generator of the linearized flow operator $L(t,A_0)$ i.e.

$$(4.13)\qquad F(t,A_0)\xi = R\xi+(1+i\nu)\xi_{xx}-2(1+i\mu)|A(t)|^2\xi-(1+i\mu)A(t)^2\xi^*$$

and $A(t)$ is the solution of the CGLE (4.2) with initial condition A_0. Let $P_n(t)$ denote the projection of X onto the span of $\xi_1(t),\ldots,\xi_n(t)$. Then since $P_n(t)\xi_i(t)$ we may rewrite the time development of the n-form $\xi_1(t)\wedge\ldots\wedge\xi_n(t)$ as

$$\begin{aligned}(4.14)\qquad (d/dt)\xi_1(t)\wedge\ldots\wedge\xi_n(t) &= F(t,A_0)\cdot P_n(t)\xi_1(t)\wedge\ldots\wedge\xi_n(t)+\ \ldots\\ &\quad +\xi_1(t)\wedge\ldots\wedge F(t,A_0)\cdot P_n(t)\xi_n(t)\ .\end{aligned}$$

The time derivative of the n-volume V_n spanned by $\xi_1(t),\ldots,\xi_n(t)$ is then given by

$$(4.15)\qquad (d/dt)V_n^{\,2} = 2V_n(d/dt)V_n = (d/dt)\|\xi_1\wedge\ldots\wedge\xi_n\|^2$$
$$= (d/dt)\langle\xi_1\wedge\ldots\wedge\xi_n,\ \xi_1\wedge\ldots\wedge\xi_n\rangle$$
$$= 2\ Re(\langle(d/dt)(\xi_1\wedge\ldots\wedge\xi_n),\ \xi_1\wedge\ldots\wedge\xi_n\rangle)$$
$$= 2\ Re(\langle F(t,A_0)\cdot P_n(t)\xi_1(t)\wedge\ldots\wedge\xi_n(t),\ \ \xi_1\wedge\ldots\wedge\xi_n\rangle + \ldots$$
$$\ldots + \langle\xi_1\wedge\ldots\wedge F(t,A_0)\cdot P_n(t)\xi_n,\ \xi_1\wedge\ldots\wedge\xi_n\rangle)$$
$$= 2\ \|\xi_1\wedge\ldots\wedge\xi_n\|^2 Re(Tr\ F(t,A_0)\cdot P_n(t))\ ,$$

where $Tr\ F(t,A_0)\cdot P_n(t)$ denotes the trace of the finite rank operator $F(t,A_0)\cdot P_n(t)$. This trace formula, i.e., the last step in (4.15) above, may be proved in general by writing the ξ_i's in terms of the orthonormal vectors spanning $P_n(t)\mathcal{H}$. It is derived in the context of Lyapunov exponents in [1], and it is trivial in the case that the ξ_i's are eigenvectors of $F(t,A_0)\cdot P_n(t)$. Solving (4.15), we obtain the time development of the volume V_n:

$$(4.16)\qquad V_n(t) = V_n(0)\ exp(Re(\int_0^t ds\ Tr\ F(s,A_0)\cdot P_n(s)))\ .$$

From equation (4.6), the sum of the first n global Lyapunov exponents may be expressed

$$(4.17)\qquad \lambda_1+\ldots+\lambda_n = \limsup_{t\to\infty} t^{-1}log[\sup_{A_0}\ \sup_{\|\xi_i\|\le 1}\ \times$$
$$\times\ \ exp(Re(\int_0^t ds\ Tr\ F(s,A_0)\cdot P_n(s)))],$$

where, as before, the supremum over A_0 is restricted to the universal attractor. We remark that although the ξ_i's are not explicitly present in (4.17) above, they enter the formula via the time dependent projection $P_n(s)$.

A lower bound on the sum of the first n global Lyapunov exponents is immediately obtained by noting that $A_0 = 0$ is contained in the universal attractor and the nonlinear solution with this initial condition is $A(t) = 0$. Thus,

$$(4.18)\qquad \sup_{A_0} \sup_{\|\xi_i\|\le 1} \exp(Re(\int_0^t ds\ Tr\ F(s,A_0)P_n(s)))$$

$$\ge \sup_{\|\xi_i\|\le 1} \exp(Re(\int_0^t ds\ Tr(s,0)\cdot P_n\ (s)))$$

$$\ge \exp(Re(\int_0^t ds\ Tr\ F(s,0)\cdot P_n')) \ ,$$

where P_n' is the projection onto the first n Fourier coefficients $\phi_j(x) = exp(ik_jx)$, in the order $(j=1)$ $k_1 = 0$, $(j=2)$ $k_2 = 2\pi$, $(j=3)$ $k_3 = -2\pi$, $(j=4)$ $k_4 = 4\pi$, $(j=5)$ $k_5 = -3\pi$, etc. The trace in the last term of (4.18) above is easily evaluated:

$$(4.19)\qquad Tr\ F(t,0)\cdot P_n' \sum_{n=1}^{n} \langle\phi_j, (R+(1+i\nu)\partial_x^{\ 2})\phi_j\rangle$$

$$= \sum_{j=1}^{n} (R-(1+i\nu)k_j^{\ 2}).$$

Hence we have the lower bound

$$(4.20)\qquad \lambda_1 + \ldots + \lambda_n \ge \sum_{j=1}^{n} (R-k_j^{\ 2}) \quad .$$

Note that this lower bound is independent of the imaginary diffusion, ν. It is also worth remarking that this estimation of the global Lyapunov exponents is closely related to the sideband stability analysis of the rotating wave solutions carried out in §3. In fact, other lower bounds on the sum of the first n global Lyapunov exponents can also be obtained by the argument above by noting that one may choose to linearize about *any* of the rotating wave solutions. For a specific choice of parameters, one may obtain lower bounds in terms of the growth rates of sideband perturbations about the rotating wave solutions. Since we do not have explicit expressions for these rates, i.e., expressions for the real parts of the eigenvalues of the matrix in (3.7), we will not pursue this idea here. It is possible, however, that these considerations could lead to sharper lower bounds, especially deep in the modulationally unstable regime.

Upper bounds on the sum of the first n global Lyapunov exponents are obtained by bounding the real part of the trace in (4.17) from above. Let ψ_j be a set of orthonormal vectors spanning $P_n(t)K$. Then

$$(4.21)\qquad Re(Tr\ F(t,A_0)\cdot P_n(t)) = Re \sum_{j=1}^{n} \langle \psi_j, F(t,A_0)\psi_j \rangle$$

$$= \sum_{j=1}^{n} [\langle \psi_j, (R+\partial_x^{\,2})\psi_j\rangle - 2\langle \psi_j, |A|^2\psi_j\rangle - Re((1+i\mu)\langle \psi_j, A^2\psi_j^{\,*}\rangle)].$$

For any vector $\psi \in X$ and any A_0 on the universal attractor,

$$(4.22)\qquad -2\langle \psi, |A|^2\psi\rangle - Re((1+i\mu)\langle \psi, A^2\psi^*\rangle) =$$

$$= -2\int_0^t dx|A|^2|\psi|^2 - Re((1+i\mu)\int dx\ A^2\psi^{*2})$$

$$\le (-2+|1+i\mu|)\|A\|_\infty^2\ \|\psi\|_2^2$$

$$\le \delta\|A\|_\infty^2\ \|\psi\|_2^2 \quad .$$

where $\delta = max(0,(-2+|1+i\mu|))$, and $\|A\|_\infty^2$ is the uniform bound on all solutions on the universal attractor (cf. equations (2.10)). Thus,

$$(4.23)\qquad Re(Tr\ F(t,A_0)\cdot P_n(t)) \le \sum_{j=1}^{n} \langle \psi_j, R+\delta\|A\|_\infty^2 + \partial_x^{\,2})\psi_j\rangle$$

$$\le \sum_{j=1}^{n} \langle \phi_j, (R+\delta\|A\|_\infty^2 + \partial_x^{\,2})\phi_j\rangle$$

$$\le \sum_{j=1}^{n} (R+\delta\|A\|_\infty^2 - k_j^{\,2}) \quad ,$$

equation (4.19). Utilizing (4 17), we obtain the upper bound

$$(4.24)\qquad \lambda_1 + \ldots + \lambda_n \le \sum_{j=1}^{n} (R\ \delta\|A\|_\infty^2 - k_j^{\,2}) \quad .$$

We first note that $\delta = 0$ when $|\mu| \le \sqrt{3}$, so the upper bound (4 24) ***coincides*** with the lower bound (4.20), thereby yielding the global Lyapunov exponents exactly! As shown in the last two sections, the parameter regime $|\mu| \le \sqrt{3}$ is very special in the CGLE. Here we have shown that in this regime, the global Lyapunov exponents are exactly those corresponding to the linearized CGLE. This regime is distinguished here, as in §2, in that the *linearization* of the *nonlinear* part of the CGLE has a negative real part as an operator on X. We also remark that the computation of the Lyapunov exponents for $|\mu| \le \sqrt{3}$ does *not* depend on the dimension of the space in which the CGLE is posed: the same formula holds for the CGLE when the spatial variable x lives in a bounded domain in R^d, with the spectrum of the *d*-dimensional Laplacian treated appropriately. Secondly, the uniformity of the dimension bounds in the imaginary diffusion appears here as it did in the last two sections in the context of the linear stability analysis and the Fourier spanning dimension. We turn now to the computation of d_L for the case $|\mu| \le \sqrt{3}$. The Lyapunov exponents are

$$(4.25) \qquad \lambda_n = R - (2\pi)^2[n/2] \quad ,$$

where, as before, the bold square brackets indicate the integer part. The Lyapunov dimension is then computed directly from the defining procedure, equations (4.8-9). A plot of d_L vs. the control parameter $R^{1/2}/2\pi$ is given in Figure 6. It is a continuous curve with a discontinuous derivative at the points where d_L is an odd integer. Additionally, we may compute an analytic upper bound on the Lyapunov dimension which is *exact* at the points where d_L is an odd integer:

$$(4.26) \qquad d_L \le 2(3R/4\pi^2+1/4)^{1/2} \quad .$$

The upper bound (4.26) is also plotted in Figure 7, as well as the lower bound on the dimension given by (2.17a). Note the uniformity in μ as well as ν in this parameter regime. The upper and lower bounds are both asymptotically proportional to $R^{1/2}$, and their slopes for large R differ only by a fctor of $\sqrt{3}$.

When $|\mu| > \sqrt{3}$, we only determine the upper bounds on d_L (and hence d_H) rather than computing it exactly. If the sum of the first n Lyapunov exponents is less than or equal to 0, then $d_L < n + 1$. From the upper bound on the sum of the first n Lyapunov exponents (4.24), using the fact that $n \le 2[(n-1)/2]+1$, we determine the a sufficient condition for $d_L < n + 1$ is

$$(4.27) \qquad (2[n-1)/2] + 1)(R+\delta\|A\|_\infty^2) - (2\pi)^2[(n-1)/2]([(n-1)/2]+1)$$
$$(2[(n-1)/2]+1)/3 \le 0 \quad .$$

From this expression it is easy to solve a quadratic equation in $[(n-1)/2]$ and find that

$$(4.28) \qquad d_L < 2(3(R+\delta\|A\|_\infty^2)/4\pi^2+1/4)^{1/2} + 1 \quad .$$

The expression (2.10) for $\|A\|_\infty^2$ may be inserted into (4.28) above to express the upper bound on d_L in terms of R and $|\mu|$ in the case $\nu \neq 0$. For large R, the upper bounds on d_H and d_L may be summarized as

$$(4.29a) \qquad d_H \le d_L < 2\sqrt{3}(R^{1/2}/2\pi) + 1, \qquad |\mu| \le \sqrt{3},\ \nu \text{ arbitrary},$$

$$(4.29b) \qquad d_H \le d_L < (\sqrt{3}/\pi)|\mu|R+3|\mu|^{1/2}R^{1/2}/2\pi + 2, \qquad |\mu| > \sqrt{3},\ \nu \text{ arbitrary}.$$

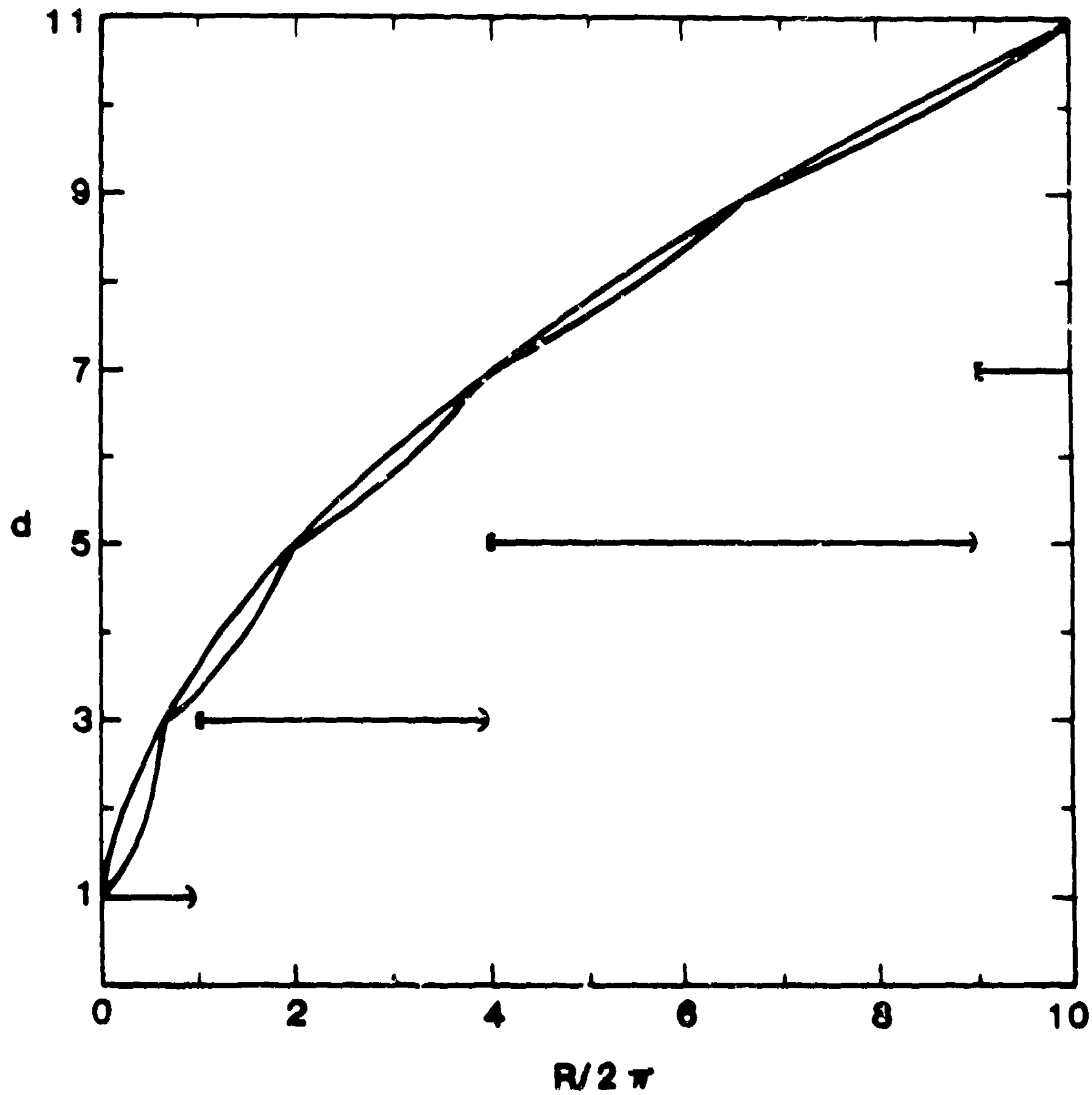

Figure 6: Plot of the Lyapunov dimension vs. effective Reynolds number for the CGLE with $|\mu| < \sqrt{3}$ (piecewise differentiable curve) and the upper bound. The piecewise constant curve is the lower bound on the attractor dimension D_{rot}.

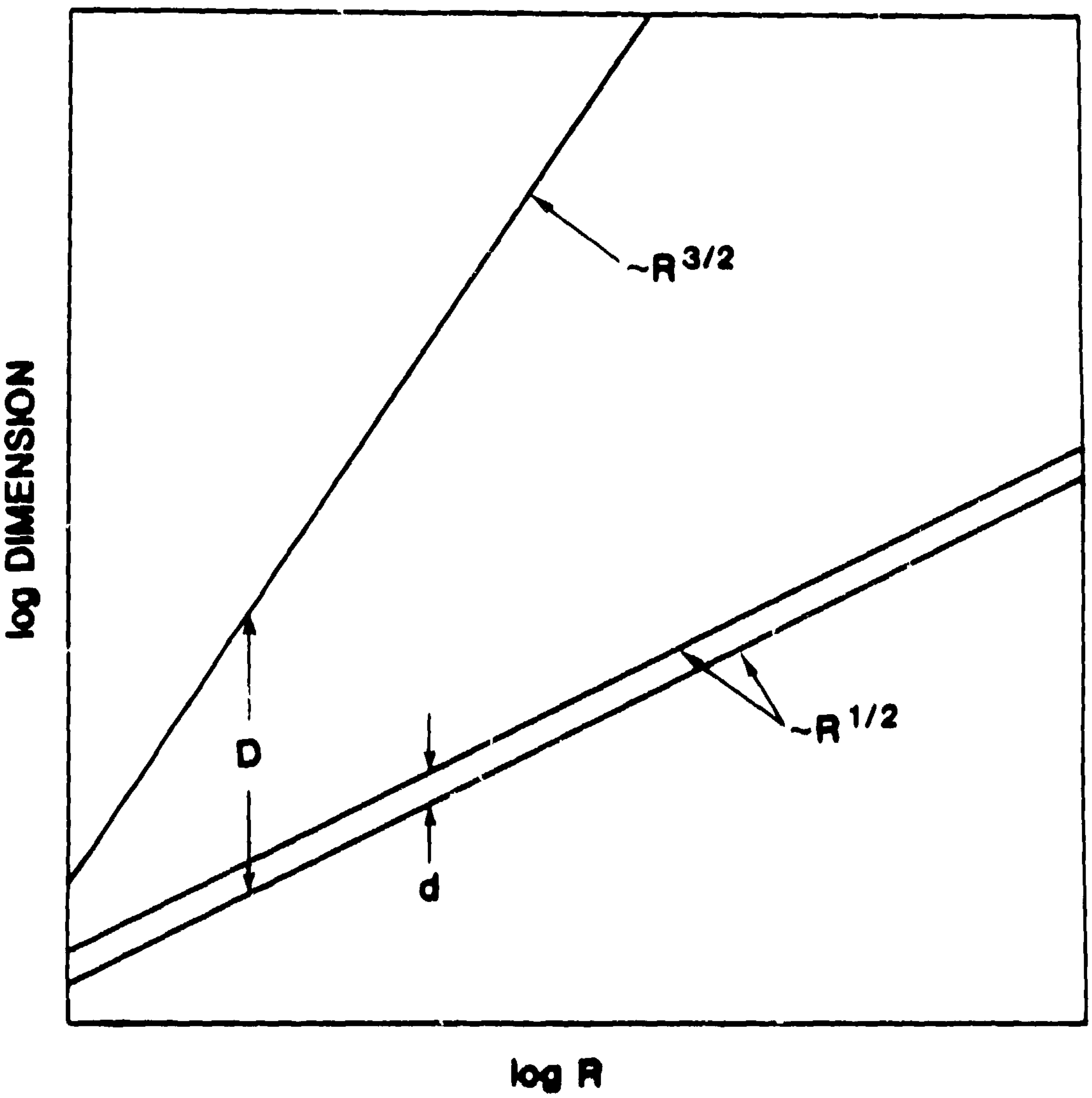

Figure 7: Upper and lower bounds on the attractor and Fourier spanning dimensions versus logR with $|\mu| < \sqrt{3}$.

Results can also be found [37] when $\nu = 0$ and μ arbitrary. Finally, we remark that in July 1987 at this same AMS meeting we discovered that J.M. Ghidaglia & B. Heron [39] have also calculated essentially the same attractor dimension estimates for CGLE as in our §4 and we thank them for a copy of their paper.

BIBLIOGRAPHY

1. P. Constantin & C. Foias, Comm. Pure Appl. Math., 38, (1985), 1.

2. P. Constantin, C. Foias, B. Nicolaenko & R. Temam, C.R. Acad. Sci. Serie 1, 302, (1986), 375-8.

3. C. Foias, B. Nicolaenko, G.R. Sell & R. Temam, C.R. Acad. Sci. Paris, Serie 1, 301, (1985), 285.

4. C. Foias, G.R. Sell & R. Temam, IMA preprint, (1986), 234.

5. C. Foias, G.R. Sell & R. Temam, C.R. Acad. Sci. Paris, Serie 1, 301, (1985), 139.

6. J. Mallet-Paret & G.R. Sell, Brown University Preprint (1986).

7. J. Mallet-Paret & G.R. Sell, Brown University Preprint (1987).

8. J. Mallet-Paret & G.R. Sell, Brown University Preprint (1987).

9. C. Foias, B. Nicolaenko, G.R. Sell & R. Temam, J. Math. Pure & Appl. to appear 1987.

10. B. Nicolaenko, B. Scheurer & R. Temam, Physica 16D, (1985), 155.

11. J.M. Hyman & B. Nicolaenko, Physica 18D, (1986), 113.

12. A.C. Newell & J.A. Whitehead, J. Fluid Mech., 38, (1969), 279.

13. R.C. DiPrima, V. Eckhaus & L.A. Segel, J. Fluid Mech, 49, (1971), 705.

14. A.C. Newell & J.A. Whitehead, Proc. IUTAM Symposium Herrenhalb 1969, Springer, Berlin, 1971.

15. S. Kogelman & R.C. DiPrima, Phys. Fluids, 13, (1970), 1.

16. K. Stewartson, & J.T. Stuart, J. Fluid Mech, 48, (1971), 529.

17. L. Hocking, K. Stewartson & J.T. Stuart, J. Fluid Mech., 51, (1972), 705.

18. L. Hocking & K. Stewartson, Proc. Roy. Soc. Lond. A326, (1972), 289

19. J. Ball, Quart. J. Math. Oxford, 28, (1977), 473.

20. P. Blennerhassett, Phil. Trans. Roy. Soc., 298, (1980), 451 , and H. Brandt, P. Lomdahl & A.C. Newell, Physica 23D, (1986), 345.

21. Y. Kuramoto & T. Yamada, Prog. Theor. Phys. Suppl., 56, (1976), 679

22. Y. Kuramoto, Prog. Theor. Phys. Suppl., 64, (1978), 346.

23. Y. Kuramoto & S. Koga, Prog. Theor. Phys. Suppl., 66, (1981), 1081

24. W. Eckhaus, Studies in Nonlinear Stability Theory, Springer, Berlin, 1965.

25. C. Lange & A.C. Newell, SIAM J. Appl. Math., 27, (1974), 441.

26. J.T. Stuart & R.C. DiPrima, Proc. Roy. Soc. Lond. A362, (1978), 27.

27. Catherine Holmes, Proc. Roy. Soc. Lond. A402, (1985), 229.

28. H.T. Moon, P. Heurre & L. Redekopp, Phys. Rev. Lett., 49, (1982), 458.

29. H.T. Moon, P. Heurre & L. Redekopp, Physica 7D, (1983), 135.

30. L. Keefe, Stud. Appl. Math., 73, (1985), 91.

31. K. Nozaki & N. Bekki, Phys. Rev. Lett., 51, (1983), 2171.

32. P. Holmes, Physica 23D, (1986), 84.

33. A.J. Bernhoff, University of Arizona, Mathematics Department Preprint 1987.

34. J. Carr, Applications of Center Manifold Theory, Springer, New York, 1981.

35. J. Kaplan & J. Yorke, in Functional Differential Equations and Approximations of Fixed Points, edited by H.O. Peitgen & H.O. Walther, Springer, Berlin, 1979.

36. C. Doering, J.D. Gibbon, D.D. Holm & B. Nicolaenko, Low Dimensionality in the Complex Ginzburg-Landau Equation, to appear in 'Nonlinearity', 2nd issue, 1988..

37. C. Doering, J.D. Gibbon, D.D. Holm & B. Nicolaenko; Exact Lyapunov dimension of the universal attractor for the CGL equation: Phys Rev. Lett., 59 (1987), 2911-14.

38. C. Doering, J.N. Elgin, J.D. Gibbon & D.D. Holm; Low dimensionality in the laser equations in the good cavity limit; preprint 1987.

39. J.M. Ghidaglia & B. Heron, Physica 28D, (1987), 282.

J.D. Gibbon
Department of Mathematics
Imperial College
London SW7 2BZ

D.D. Holm & B. Nicolaenko
T. Division (T7)
Los Alamos National Lab.
New Mexico 87545

C. Doering
Dept of Physics
Clarkson Univ.
Potsdam, N.Y.